Atti Accademia Peloritana dei Pericolanti
Classe I di Scienze Fis. Mat. e Nat.
Vol. LXXII (1994) - Conferenza del 2 maggio 1994

DYNAMIC DEGREES OF FREEDOM AND THE THEORY OF PLASTICITY

J. VERHÁS

The attempts to describe plastic flow within the framework of non-equilibrium thermodynamics can be traced back for long. Non-linear constitutive equations can display plastic properties even in the state of local equilibrium. The yield condition may be built into the entropy function or into the constitutive equations. A theory based on the hypothesis of local equilibrium has a lot of advantages but the properties of the admissible models are not flexible enough to give quantitative account on real materials. The need to describe several kinds of structural materials makes the introduction of dynamica degrees of freedom indispensable. The dynamic degrees of freedom give rise to a number of new problems not present in the classical theories. Firs of all, their physical meaning is a great challenge. The possible topological structure of the thermodynamic state space is also problematic. Nevertheless, the above problems are living, the advantages worth a lot.

I intend to survey some methods plastic flow can be described by within the framework of irreversible thermodynamics.

1. Systems in local equilibrium.

First a medium is assumed the local state of which is given by the internal energy and the specific volume. For the sake of simplicity, volume changes will not be regarded; the specific volume is rather a parameter than a variable. The balance equation for the internal energy is of the usual form

$$\rho \frac{du}{dt} + \operatorname{div} J_q = t : \overset{\circ}{d} \tag{1}$$

where ρ is the density; u the specific internal energy; J_q the heat flow; t Cauchy's stress; $\overset{\circ}{d}$ the strain rate, i.e. the symmetric part of the velocity gradient. The specific entropy is supposed to depend on only the internal energy

$$s = s(u).$$

Here $s(u)$ is the equilibrium entropy function. The balance equation for the entropy reads

$$\rho \frac{ds}{dt} + \operatorname{div} J_s = \sigma_s, \tag{2}$$

where J_s is the entropy flow connected to the heat flow the usual way;

$$J_s = \frac{1}{T} J_q.$$

The entropy production density is

$$\sigma_s = \frac{1}{T} t : \overset{\circ}{d} + J_q \operatorname{grad} \frac{1}{T}. \tag{3}$$

Not intending to deal with heat conduction, isotherm circumstances are presumed; so the energy dissipation function reduces to

$$T\sigma_s = t : \overset{\circ}{d}. \tag{4}$$

If Onsager's linear laws were valid the equations of Newtonian fluids would be obtained, which read

$$t = 2\eta \overset{\circ}{d}.$$

For plastic flow, the conducvtivity coefficient denoted here by 2η turn to be a fucntion depending on the thermodynamic force $\overset{\circ}{d}$. A proper choice for the fucntion leads to a theory of plastic flow. If the fucntion is chosen as

$$2\eta = \frac{k\sqrt{2}}{\sqrt{\overset{\circ}{d} : \overset{\circ}{d}}}$$

we arrive at Mieses' theory of plasticity. The model obatined this way is the ideally plastic body, the constitutive equation of which has singularity at equilibrium. Nevertheless, the ideally plastic body is suitable for practical applications, the above mentioned singularity is annoying. The constitutive equation of Binghm's body is obtained by taking

$$2\eta = 2\eta_0 + \frac{k\sqrt{2}}{\sqrt{\overset{\circ}{d} : \overset{\circ}{d}}}.$$

The singularity is also present. One may suspect that the mentioned theories are only approximations not valid for very slow motions. The linear equations of thermodynamics for media with dynamic degrees of freedom are free of the singularity.

2. The stability of viscoelastic flow.

Next a medium with several dynamic degreed of freedom are discussed. The non-equilibrium state variables are presumed to be of α-type, i.e. they are invariant under time inversion. The balance equation for the internal energy is the same as above. The entropy is very different. The non-equilibrium states are

assumed to be more complicated than the local equilibrium ones, to give account on the non-equilibrium structures developing during irreversible processes, several – say n – sy trace, second order tensors are needed. According to the Morse-lemma, the independent variables may be chosen so that the entropy has the form

$$s = s^e\left(u - \frac{1}{2\rho}\sum_i \alpha_i : \alpha_i\right), \tag{5}$$

where $s^e(u)$ is the equilibrium entropy function. The balance equation of the entropy is equation (2) and the entropy flow is taken in the classical form. The actual expression for the energy dissipation function reads

$$T\sigma_s = t : \mathring{d} - \sum_i \alpha_i : \mathring{\alpha}_i. \tag{6}$$

For simplicity, isotherm motion is presumed. Here the small circle above the symbol of a quantity denotes the co-rotational time derivative. The co-rotational time derivative has the advantage that it is a time derivative in a Euclidean frame and it is objective; so it is zero if the change of the quantity is due to rotation. The linear Onsager equations are

$$\begin{aligned} t &= L_{00}\mathring{d} - \sum_i L_{0i}\alpha_i \\ \mathring{\alpha}_i &= -L_{0i}\mathring{d} - L_i\alpha_i. \end{aligned} \tag{7}$$

Here the fact that an orthogonal transformation preserves the form of the entropy transformation preserves the form of the entropy function (5) has been utilized. This way the relaxation of each dynamic variable is independent from any other.

The solutions of the equations are interesting and display complicated behavior. For steady shear flow the viscometric

functions have the forms

$$\tau(\chi) = \frac{L_{00}}{2}\chi + \sum_i \frac{L_{0i}L_i}{2}\frac{\chi}{L_i^2+\chi^2};$$
$$\sigma_1(\chi) = -\sigma_2(\chi) = \sum_i \frac{L_{oi}}{2}\frac{\chi^2}{L_i^2+\chi^2}. \tag{8}$$

Here χ is the rate of shear. The shear stress function may have decreasing sections where the solution is unstable. The graph of the function is shown in figure 1 for a body with one dynamic degree of freedom.

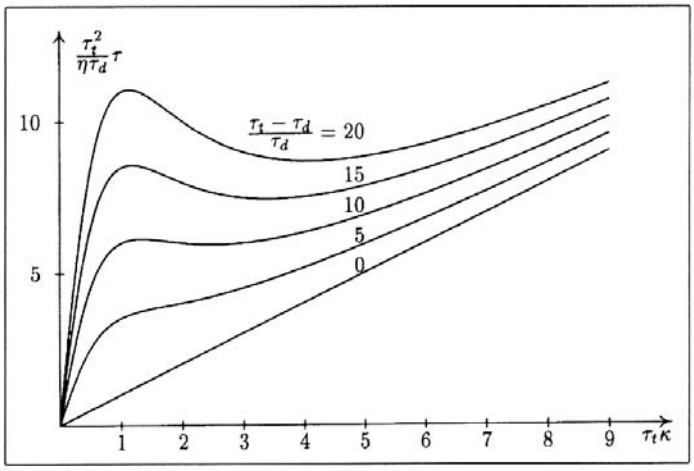

Figure 1

What happens if we start the shear flow? We can apply some shear stress regulated. The average shear rate may be also regulated but the local shear rate. So the increasing stress causes slow flow up to a critical stress. If the stress overrides the critical one the shear rate jumps to a higher value but not at places. The equations do not determine where the jump happens. The distribution of high shear rate regions may be

random. If the pattern of the high and low shear rate regions is fine an averaging over them is obvious. These considerations have us incline to apply a non-linear theory to give account on the average properties.

The investigation of the uniaxial load also shows interesting features. A solution

$$t = 2\eta \overset{\circ}{d} \tag{9}$$

exists at any load but the analysis of the stability shows that at high load the above solution is unstable. The elongational flow collapses and rotation emerges with angular velocity perpendicular to the traction. The rateof elongation increases rapidly with increasing load. The direction of the angular velocity is not determined by the equations; the direction may be random. If the pattern of the domains of uniform rotation is fine the averaging is again obvious; a non-linear theory is again motivated.

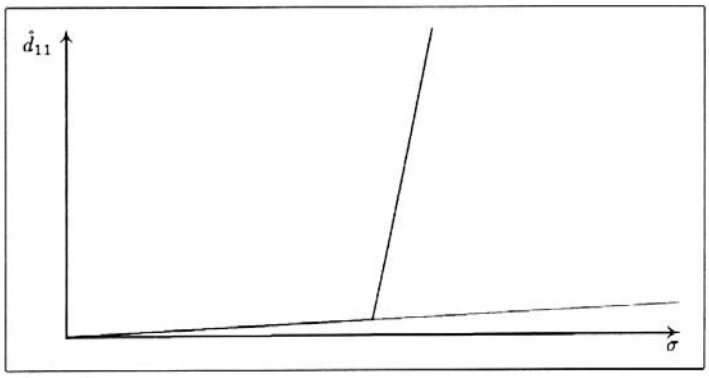

Figure 2

The rate of elongation plotted against the tension is shown in figure 2 for a material with one dynamic degree of freedom.

3. Generalized constituive equations.

Next a non-linear theory will be presented. The model is similar the one above only the linear constitutive equations (7) are modified "slightly". The second term on the right hand side of the relaxation equations is replaced by a non-linear fucntion.

$$
\begin{aligned}
t &= L_{00}\overset{\circ}{d} - \sum_i L_{0i}\alpha_i \\
\overset{\circ}{\alpha}_i &= L_{0i}\overset{\circ}{d} - f_i(\alpha_i).
\end{aligned}
\tag{10}
$$

The actual $f_i(\alpha_i)$ functions have to be found out. According to the traditional theories of plasticity, the f_i functions – they are isotropic functions – are taken in the form

$$f_i = \lambda_i(\alpha_i)\alpha_i,$$

where the values λ_i are scalar and depend on the invariants of the arguments. The analogy to von Mieses' theory suggests that λ_i depends on the quantity $\sqrt{\alpha_i : \alpha_i}$. To arrive at a theory of plastic flow the function has to have very small value if the argument is smaller than the one belonging to the yield condition and has to increase rapidly after. The chemical reaction kinetic proposes exponential functions like

$$y = A \sin h(\chi x).$$

With extreme parameters A and χ the graph of the function looks like in figure 3. In such a case the constitutive equation for the dynamic variable α_i may be replaced by the correlation according to which $\lambda_i(\alpha_i)$ is zero if α_i is under its own yield condition and arbitrary if α_i satisfies it. One can see that the theory obtained gives the model for plastic flow. The rheological model consists of n Prandtl-Reuss element and a Newton element connected in parallel; the latter is usually

dropped (figure 4). To get this picture easier, replace the dynamic variable α_i by

$$\alpha_i = -\frac{1}{L_{0i}} t_i .$$

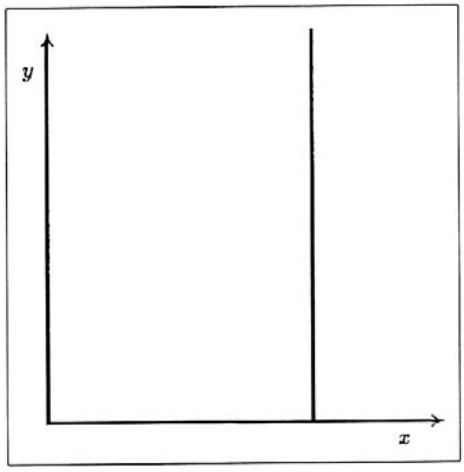

Figure 3

For the new variables, the set of the constitutive equations takes the form

$$t = L_{00}\overset{\circ}{d} + \sum_i t_i$$

$$\dot{t}_i = L_{0i}^2 \overset{\circ}{d} + g_i$$

where g_i is zero if t_i is under the yield condition and arbitrary is t_i satisfies it. Here the stress is split into parts. Each part has its own yield condition. If g_i is zero the part of the stress is elastic ($\dot{t}_i = L_{0i}^2 \dot{d}$); if not, the material flows. A model with sufficiently large n describes the experiments well.

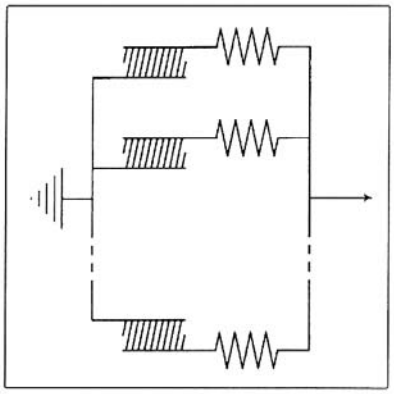

Figure 4

Acknowledgement.

Work has been supported by G.N.F.M. of C.N.R., M.U.R.S.T. (40% and 60%), O.T.K.A. (50%, 30%, and 20%) and Italy-Hungary TET II/2.

REFERENCES

Bampi F., Morro A., *Objectivity and Objective Time Derivativesin Continuum Physics.* Foundations of Physics, **10**, (1980) 905-920.

Bhattacharya D. K., *On the Problem of Evolution Criterion for Thermodynamical Waves.* Acta Mechanics **47**, (1983) 97-94.

Casimir H. G. B., *On the Onsager's Principle of Microscopic Reversibility.* Rev. Mod. Phys. **17**, (1945) 343-350.

Ciancio V., Kluitenberg G. A., *On Linear Dynamical Equations of State for Isotropic Media II.* Some Cases of Special Interest. Physica **99A** 592-600.

Ciancio V., Restuccia L., *Asymptotic Waves in Anelastic Media without Memory (MAxwell Media).* Physica **131A** (1985) 251-262.

Ciancio V., Restuccia L., *Non-Linear Dissipative Waves in Viscoelastic Media.* Physica **132A** (1985) 606-616.

Ciancio V., Restuccia L., *The Generalized Burgers Equation in Viscoanalastic Media With Memory.* Physica **142** (1987) 309-320.

Ciancio V., Verhás J., *Plasticity as a Consequence of the Instability of Viscoelastyc Flow.* A Thermodynamic Approach. Le Matematiche, **46** (1991) 77-83.

Cox W. P., Merz E. H., *Correlation of Dynamic and Steady-Flow Viscosities.* J. Polym. Sci. **28** (1958) 619-622.

Cox W. P., Merz E. H., *Rheology of Polymer Melts.* A Correlation of Dynamic and Steady-Flow Measurement. Am. Soc. Testing Material Spec. Techn. Publ. **247**, (1958) 178-188.

De Groot S. R., *Thermodynamics of Irreversible Processes.* North-Holland Publ. Co. Amsterdam 1951.

De Groot S. R., Mazur P. *Non-equilibrium Thermodynamics.* North-Holland Publ. Co. Amsterdam 1962.

drouod R., Maugin G. A., *Thermodynamical Modelling of Polymers in Solution. In Constitutive Laws and Microstructure.* eds D. R. Axelrad and W. Muschik (1988) 136-161, Springer, Berlin.

Edelen D. G., *A Thermodynamic Derivation of Non-history Dependent Constitutive Relation for Elastic, Viscoelastic, Fluid and Perfectly Plastic Bodies.* Arch. of Mech. **26** (1974) 251-261.

Edelen D. G. B., *Generaliesd Onsager Fluxes and Forces.* A Nonlinear Phenomenological Theory. Z. f. Phys. Chem. **80** 37-53.

Edelen D.G. B., *On the Characterization of Fluxes in Nonlinear Irreversible Thermodynamics.* Int. J. Engng. Sci. **12**, (1974) 397-411.

Edelen D. G. B., *A Thermodynamics with Internal Degrees of Freedoms and Nonconservative Forces.* Int. J. Engng. Sci. **14** (1976), 1013-1032.

Eringen A. C., *Irreversible Thermodynamics and Continuum Mechanics.* Phys. Rev. **117** (1960) 1174-1183.

Frias M., *Thermodynamically Structured Flows.* Rheology III. 749. Plenum Press. New York 1980.

Gambár K., Márkus F., *On the Global symmetry of Thermodynamics and Onsager's Recoprocity Relations.* J. Non-Equilib. Thermodyn. **18** (1993) 51-57.

Gambár K., Máckus F., Nyíri B., *A Variational Principle for the Balance and Constitutive Equations in Convective Systems.* J. Non-Equilibr. Thermodyn. **16** (1991) 217-224.

Garcia-Colin L. S., Uribe F. J., *Extended Irreversible Thermodynamics beyond the Linear Regime.* A Critical Overview. J. Non-Equilb. Thermodyn. **16** (1991) 89-128.

Garrod C., Hurley J., *Symmetry Relations for the Conductivity Tensor.*

Phys. Rev A **27** (1983) 1487-1490.

Glasstone, Laidler, Eyring, *The theory of Rate Processes.* McGraw-Hill, New York 1947.

Gyarmati I., *Non-Equilibrium Thermodynamics.* Springer, Berlin, 1970.

Gyarmati I., *On the Wave Approach of Thermodynamics and some Problems of Non-Linear Theories.* J. Non-Equilibr. Thermodyn. **2** (1977) 233-260.

Hurley J., Garrod C., *Generalization of the Onsager Reciprocity Theorem.* Phys. Rev. Lett. **48** (1982) 1575-1577.

Jou D., Casa-Vasquez J., Lebon G., *Extended Irreversible Thermodynamics: An Overview of Recent Bibliography.* J. Non-Equilib. Thermodyn. **17** (1991) 383-391.

Keller J. U., *Thermodynamik der Irreversiblen Prozesse.* W. de Guyter Berlin, New York 1977.

Kluitenberg G. A., *Thermodynamical Theory of Elasticity and Plasticity.* Physica **28** (1962) 2317-232.

Kluitenberg G. A., *On the Thermodynamics of Viscosity and Plasticity.* Physica **29** (1963) 633-652.

Kluitenberg G. A., *A Thermodynamic Derivation of the Stress-Strain Relations for Burgers Media and Related Substances.* Physica **38**, (1968) 513-548.

Kluitenberg G. A., *A Thermodynamic Discussion of the Possibility of Singular Yield Conditions in Plasticity Theory.* Physica **88A** (1977) 122-134.

Kluitenberg G. A., Ciancio V., *The Stress-Strain-Temperature Relation for Anisotropic Kelvin-Voigt Media.* Atti. Acad. di Sci. Lett. ed Arti di Palermo, serie 4, vol. XL, fasc. 2. Parte 1 (a.a; 1980-81), 107-122.

Lebon G., *From Classical Irreversible Thermodynamics to Extended Thermodynamics.* Acta Physica Hungarica **66** (1989) 241-253.

Lebon G., Perzyna P., *Recent development in thermomechanics of solids,* Springer, Wien 1980.

Lengyel S., *On Classical Nonequilibrium Thermodynamics and its Extensions.* in Recent Development in Nonequilibrium Thermodynamics. Lecturenotes in Physics 199, Springer, Berlin, Heidelberg, New York. 1984.

Lengyel S., *Deducton of the Guldberg-Waage Mass Action Law from Gyarmati's Governing Principle of Dissipative Processes.* J. Chem. Phys. **88** (1988) 1617-1621.

Lengyel S., Gyarmati I., *On the Thermodynamics of Elementary Chemical Reactions in Homogeneous Systems.* J. Chem. Phys. **75** (1981)

2384-2389.

Machlup S., Onsager L., *Fluctuations and Irreversible Processes II.* Systems with Kinetic Energy. Phys. Rev. **91** (1953) 1512-1515.

Márkus F., Gambár K., Comments *On The Relationship Between Extended Thermodynamics and the Wave Approach to Thermodynamics*, J. Non-Equilib. Thermodyn. **14** (1989) 355-362.

Márkus F., Gambár K., *A Variational Principle in Thermodynamics,* J. Non-Equilib. Thermodyn. **16** (1991) 27-32.

Márkus F., Gambár K., *Hamilton's Canonical Equation s and the Entropy Production.* J. Non-Equilib. Thermogyn. **18** (1993) 288-292.

Maugin G. A., *The Notion of Internal Variables in Fluid Mechanics* (Complex Fluids and Structurally Complex Flows). In Theoretical and Applied Mechanics, 671-676 publ. Bulgarian Acad. of Sci., Sofia, 1981.

Maugin G. A., *Internal Variables and Dissipative Structures.* J. Non-Equilib. Thermodyn. **15** (1990) 173-192.

Maugin G. A., *Thermomechanics of Plasticity and Fracture.* Cambridge University Press 1992.

Maugin G. A., Drout R., *Internal Variables and the Thermodynamics of Macromolecule Solutions.* Int. J. Engng. Sci **21** (1983) 705-724.

Meixner J., Reik H. G., *Thermodynamik der Irreversiblen Processe.* in Handbuch der Physik, Vol. III/2. 413. Springer, Berlin, 1959.

Morse M., *Relation Between the Critical Points of a Real Function of n Independent Variables.* Trans. Am. Math. Soc. **27** (1925) 345-396.

Müller I., *Thermodynamics.* Pitman Publ. Co., London 1985.

Muschik W., *Internal Variables in Non-Equilibrium Thermodynamics.* J. Non-Equilib. Thermodyn. **15** (1991) 127-137.

Muschik W., Szer J., *On the Formulation of Variational Principles of Irreversible Thermodynamics in Presence of a Magnetic Field.* J. Non-Equilib. Thermodyn. **1** (1976) 61-65.

Nyiri B., *Thermodynamical Derivation of Equations of Motion for Multicomponent Fluids.* Acta Phys. Hung. **60** (1986) 245-260.

Nyiri B., *On the Construction of Potentials and Variational Principles in Thermodynamics and Physics,* J. Non-Equilibr. Thermodyn. **16** (1991) 39-56.

Nyiri B., *On nthe Entropy Current.* J. Non-Equilib. Thermodyn. **16** (1991) 179-186.

Nyiri B., *Notes on Local Equilibrium in High Pressure Discharges.* Invited lectur, Sixth International Symposium on the Science and Technology of Light Sources, B udapest, szeptr.

Oláh K., *Thermostatics,* Thermodynamics and Thermokinetics. Acta Chim.

Hung **125** (1988) 117-130.

Oláh K., Bódiss J., Farkas H., *Thermodynamic Charges.* Acta Chim. Hung. **127** (1990) 783-794.

Onsager L., *Reciprocal Relations in Irreversible Processes I,* Phys. Rev. **37** (1931) 405-426.

Onsager L., *Reciprocal Relations in Irreversible Processes II,* Phys. Rev. **38** (1931) 2265-2279.

Onsager L., Maclup S., *Fluctuations and Irreversible Processes,* Phys. Rev. **91** 1505-1512.

Papenfuss C., Ellinghaus R., Muschik W., *Derivation of a Differential Equation for a Cholesteric Liquid Crystal in Electric Fields* Mol. Cryst. Liq. Cryst. **192** (1990) 329-337.

Parkus H., *Variational Principles in Thermo- and Magneto-Elasticity.* Springer Verlag, Wien, (1972) Wien, New York.

Perzyna P., Plochocki Z., Valanis K. C., *Thermodynamical Problems of Continuous Media.* Ossolineum, Warszawa, 1970.

Prigogine I., In *Fast Reactions and Primary Processes in Chemical Kinetics.* (ed. Claesson, S) p. 361. Stockholm, Almqvist and Wiksell; New York, Interscience 1967.

Prigogine I., Mazur P., *Sur l'extension de la thermodynamique aux phenomenes irreversibles lies aux degres de liberte internes.* Physica (Paris) **19** (1953) 241-254.

Reiner M., *Deformation, Strain and Flow.* Interscience, New York. Schapery, R. A. 1964. Application of Thermodynamics to Thermomechanical, Fracture, and Birefringent Phenomenon in Viscoelastic Media. J. Appl. Phys. **35**, (1960) 1451-1465.

Schapery R. A., *On the Time Dependence of Viscoelastic Variational Solutions.* QZ. Appl. Math. **22** (1964) 207-215.

Schowalter W. R., *Mechanics of Non-Newtonian Fluids.* Pergamon Press, New York, 1978.

Singh P., *An Approximate Technique to Thermohydrodynamic Stability Problem on the Basis of the Governing Principle of Dissipative Processes.* J. Non-Equilib. Thermodyn. **1** (1976) 105-116.

Smith G. F., Rivlin R. S., *The Anisotropic tensors.* Quart. Appl. Math. **15** (1957) 308-314.

Trostel R., Muschik W., *On the Derivation of a Passivity Condition by Methods of Statistical Physics,* J. Non-Equilib. Thermodyn. **10** (1985) 105-118.

Truesdell C., *Rational Thermodynamics,* McGraw-Hill, New York 1969.

Tung N. H., *An Application of the Governing Principle of Dissipative*

Processes (GPDP) to a Quasi-Linear Problem of Hydro-Thermodynamics. J. Non-Equilib. Thermodyn. **14** (1989) 61-67.

Turrisi E., Cinacio V., Kluitenberg G. A., *On the Propagation of Linear Transverse Acoustic Waves...* II. Some Cases of Special Interest (Polynting-Thomson, Jeffrey, Maxwell, Kelvin-Voigt, Hooke and Newton Media). Physica **116A** (1982) 594-603.

Vakulenko A. A., *On the connection of stress and strain in anelastic media.* (in russian) Dokl. Akad. Nauk. SU. **118** (1958) 665-668.

Vakulenko A. A., *Thermodynamic investigation of the connection between stress and strain in isotropic visco-plastic media.* (in russian) Dokl. Akad. Nauk. SU. **126** (1959) 736-739.

Valanis K. C., *Irreversible Thermodynamics of Continuous Media.* Springer, Wien, 1971.

Ván P., Ruszin É., *Osznovoj princip gyisszipatyivnüh processzov tyecsenyije Gartmana.* Ukranszkij Zsurnal. **36/5** (1991), 743-750.

Verkás J., *The Construction of Dissipation Potentials for Non-Linear Problems and the Application of Gyarmati's Principle to Plastc Flow.* Z. f. Phys. Chemie, **249** (1972) 119-122.

Verkás J., *On the Thermodynamic Theory of Deformation and Flow.* Periodica Polytechnica, Ser. Chem. Engng. (Budapest) **21** (1977) 319-332.

Verkás J., *On the Viscosity of Colloids.* Periodica Polytechnica Ser. Chem. Engng. (Budapest), **25** (1981) 53-61.

Verkás J., *A Thermodynamic Approach to Viscoelasticity and Plasticity.* Acta Mechanica, **53**, (1984) 125-139.

Verkás J., *Irreversible Thermodynamics for the Rheological Properties of Colloids.* Int. J. Heat Mass Transfer. **30** (1987) 1001-1006.

Verkás J., *Thermodynamics of processes in deforming media.* Atti Accademia Peloritana dei Pericolanti, **67** (191) 65-72.

Verkás J., *Thermodynamic Theory of Non-Newtonian Flow.* J. Non-Equilib. Thermodyn. **18** (1993) 311-335.

Verkás J., Papenfuss C., Muschik W., *A Simplified Thermodynamic Theory for Biaxial Nematics Using Gyarmati's Principle.* Mol. Cryst. Liq. cryst. **215** (1992) 313-321.

Ziegler H., *An Attempt to Generalize Onsager's Principle, adn its Significance for Rheological Problems.* J. Appl. Math. Phys. (ZAMP). **9**, (1958) 748-763.

Technical University,
Budapest, Hungary, H-1521